Favourite Problems, Session 1

UK - Hungary IMO Training Camp, 28th December 2012

1. *(Vera Schultz)* Let u, v, w be positive real numbers such that $u\sqrt{vw} + v\sqrt{wv} + w\sqrt{uv} \geq 1$. Find the smallest value of $u + v + w$.

2. *(Zsuzsanna Bőzse)* The following polynomial is given with missing coefficients (except the coefficient of x^{10} and 1)
$$f(x) = x^{10} + \_\, x^9 + \cdots + \_\, x^2 + \_\, x + 1$$
A and B play a game. They write real coefficients into the gaps one after another, until every term has a coefficient. A starts the game. A wins if the polynomial they get, in the end, has no real roots, B wins if it has some. Who can win with a proper strategy?

3. *(Tamás Zilahi)* For any positive integer m, consider all finite sequences of integers:
$m = a_1 < a_2 < \cdots < a_n$ where the $a_1 \cdot a_2 \cdot a_3 \cdots a_n$ product is a perfect square. Let $S(m)$ denote the smallest possible value of a_n in a such sequence. For example, $S(2) = 6$, $S(3) = 8$ and $S(4) = 4$. Prove that the sequence $S(2), S(3), S(4), \ldots$ contains every composite number exactly once.

4. *(Róbert Nagy)* We start with the sequence $S = (a, b, c, d)$ of all positive integers. We get to S_1 if we take $(|a - b|, |b - c|, |c - d|, |d - a|)$. Continuing it the same way, we get $S_2, S_3, \ldots$ Prove that after a while $S_n = (0, 0, 0, 0)$.

5. *(Márton Havasi)* Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be two distinct collections of n positive integers, where each collection may contain repetitions. If the two collections of integers $a_i + a_j$ $(1 \leq i < j \leq n)$ and $b_i + b_j$ $(1 \leq i < j \leq n)$ are the same, then show that n is a power of 2.

Favourite Problems, Session 2

UK - Hungary IMO Training Camp, 30th December 2012

1. *(Márton Havasi)* Express $\dfrac{\cos 1^\circ + \cos 2^\circ + \cdots + \cos 44^\circ}{\sin 1^\circ + \sin 2^\circ + \cdots + \sin 44^\circ}$ in the form $a + b\sqrt{c}$, where a, b and c are integers.

2. *(Balázs Maga)* Let p be a prime number such that $p \equiv 1 \mod 3$. Prove that there is a solution of the congruency $a^2 + ab + b^2 \equiv 0 \mod p$ where $0 < a, b < \sqrt{p}$ integers.

3. *(Péter Simon)* A finite number of coins are placed on an infinite row of squares. A sequence of moves is performed as follows: at each stage a square containing more than one coin is chosen. Two coins are taken from this square; one of them is placed on the square immediately to the left while the other is placed on the square immediately to the right of the chosen square. The sequence terminates if at some point there is at most one coin on each square. Given some initial configuration, show that any legal sequence of moves will terminate after the same number of steps and with the same final configuration.

4. *(Tamás Zilahi)* Let $a, b, c, d > 0$ and
$$\frac{1}{a^4+1} + \frac{1}{b^4+1} + \frac{1}{c^4+1} + \frac{1}{d^4+1} = 1$$
Prove that $abcd \geq 3$.

5. *(Viktória Fonyó)* A circle touches the circumcircle of the triangle ABC, and its sides AB and AC in D, E and G, respectively. Prove, that the bisector of $\angle BDC$ passes through the midpoint of EG.

6. *(bonus: from* KöMaL *via Zoltán Gyenes)* The names of 100 convicts are placed in 100 numbered drawers in some order. Then the convicts are brought one by one from their cells in a random order, and each of them is allowed to pull out 50 drawers one by one. If he finds his own name in a drawer then he is led to a separate room. Otherwise all 100 convicts are executed immediately. Finally, if each of them succeeds, they are all set free. Show that, knowing the rules, the prisoners can follow a strategy that results in a larger than 30% probability of going free.

Favourite Problems, Session 3

UK - Hungary IMO Training Camp, 1st January 2013

1. *(Márk Di Giovanni)* Prove the following theorem of P. Erdős and J. Surányi:
   Any positive integer k can be expressed in the form $k = \pm 1^2 \pm 2^2 \pm \cdots \pm m^2$, choosing approprite sign for each $\pm$. It can be done in infinitely many different ways for any fixed k.

2. *(József Herczeg)* Prove that if for any three points A, B, C of a polygon K exists a point M such that the segments AM, BM and CM are all inside K, then there is a point O of K such that for any point X of K the segment OX is inside of K (Krasnosselsky's theorem).

3. *(Bálint Homonnay)* Let $f(n)$ be a function defined on the set of positive integers and having all its values in positive integers as well. Prove that if $f(n+1) > f(f(n))$ for every postive integer n, then $f(n) = n$ for every positive integer n.

4. *(Róbert Nagy)* At the Secret Convention of Logicians, the Master Logician placed a band on each attendee's head, such that everyone else could see it but the persons themselves could not. There were many, many different colours of band. The Logicians all sat in a circle, and the Master instructed them that a bell was to be rung in the forest at regular intervals: at the moment when a Logician knew the colour on his own forehead, he was to leave at the next bell. Anyone who left at the wrong bell was clearly not a true Logician but an evil infiltrator and would be thrown out of the Convention post haste; but the Master reassures the group by stating that the puzzle would not be impossible for anybody present. How did they do it?

5. *(Olivér Janzer)* For a, b, c, d positive real numbers $a + b + c + d = abc + abd + acd + bcd$ holds. Prove that

$$(a+b)(c+d) + (a+d)(b+c) \geq 4\sqrt{(1+ac)(1+bd)}$$

Favourite Problems, Session 4

UK - Hungary IMO Training Camp, 3rd January 2013

1. *(Ágnes Kúsz)* Each cell of an $n \times k$ array is coloured black or white. In a step we recolour the cells, according to the following rule: if a square has an odd number of black neighbours (in the old colouring), it will be black (in the new colouring); else it will be white.
   For a fixed colouring we know that after a finite series of steps, we get it back. Prove the following: if we count the number of black unit squares after each step of this finite series and add these numbers up, we get an even number.

2. *(Ádám Sagmeister)* Let A denote the set of all sequences $\{a_t : t \in \mathbb{N} \cup \{0\}\}$ that satisfy the equation $a_{m+k}a_n - a_{n+k}a_m = a_k a_{n-m}$ for all k, m, n nonnegative integers. Prove that for any chosen nonnegative integer r there exists a polynomial p_r such that $a_1 p_r(a_2/a_1) = a_r$ for any sequence $\{a_t\}$ in A.

3. *(Zsombor Fehér)* Show an example of a sequence of positive real numbers $a_1, a_2, \ldots, a_N$ which satisfies the following property: for any positive integers $1 = n_0 < n_1 < \ldots n_k = N$ the inequality below holds.

$$n_1 a_{n_0} + n_2 a_{n_1} + \cdots + n_k a_{n_{k-1}} > 2.7(a_1 + a_2 + \cdots + a_N)$$

4. *(Barnabás Janzer)* Prove that if a, b, c are real numbers such that $a^2 + b^2 + c^2 = 1$, then:

$$a + b + c < 2abc + \sqrt{2}$$

5. *(Péter Ágoston)* Let G be a graph with n vertices, whose diameter is 2 and has no vertices with a degree of $n-1$. Suppose that if the vertices are $v_1, v_2, v_3, \ldots, v_n$ and the degree of v_i is a_i, then $\sum_{i=1}^{n} a_i^2 = n^2 - n$. Let k be the length of the shortest circle in G. Determine all possible values of k.