IPS 1. - 28.12.2012.

**1.** Two circles intersect at points $A$ and $B$. A common tangent touches the first circle at point $C$ and the second at point $D$. Let $\angle CBD > \angle CAD$. Let the line $CB$ intersect the second circle again at point $E$. Prove that $AD$ bisects the angle $CAE$.

**2.** Prove the inequality where $a, b, c$ are positive real numbers.

$$\frac{a^3}{a^2+ab+b^2} + \frac{b^3}{b^2+bc+c^2} + \frac{c^3}{c^2+ca+a^2} \geq \frac{a+b+c}{3}$$

**3.** Let $a$ and $b$ be positive integers. Show that if $4ab-1$ divides $(4a^2-1)^2$, then $a=b$.

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IPS 2. - 29.12.12.

**1.** All expressions of the form

$$\pm\sqrt{1} \pm \sqrt{2} \pm \sqrt{3} \pm \cdots \pm \sqrt{100}$$

(with every possible combination of the signs) are multiplied together. Prove that the result is (a) an integer; (b) the square of an integer.

**2.** Let $P$ be a point inside the triangle $ABC$. The lines $AP$, $BP$ and $CP$ intersect the circumcircle $\Gamma$ of triangle $ABC$ again at the points $K$, $L$ and $M$ respectively. The tangent to $\Gamma$ at $C$ intersects the line $AB$ at $S$. Suppose that $SC = SP$. Prove that $MK = ML$.

**3.** Consider an $n \times n$ board, where $n$ is even. The board is divided into $n^2$ unit squares. We say that two different squares are adjacent if they have a common side. $N$ unit squares are marked in such a way, that every square (marked or unmarked) on the board is adjacent to at least one marked square. Determine the smallest value of $N$.

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IPS 3. - 30.12.12.

**1.** Prove that the equation has infinitely many solutions in integers $x$, $y$ and $z$.

$$xy(x-y) + yz(y-z) + zx(z-x) = 6$$

**2.** Determine all pairs $(n;p)$ of positive integers such that $p$ is a prime, $n \leq p$ and $(p-1)^n+1$ is divisible by $n^{p-1}$.

**3.** Let $I$ be the incentre of triangle $ABC$ and let $\Gamma$ be its circumcircle. Let the line $AI$ intersect $\Gamma$ again at $D$. Let $E$ be a point on the arc $\widehat{BDC}$ and $F$ a point on the side BC such that

$$\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$$

Finally, let $G$ be the midpoint of the segment $IF$. Prove that the lines $DG$ and $EI$ intersect on $\Gamma$.

Tata test - 31.12.12

**1.** (a) The numbers $1, 2, ..., 100$ are divided into two groups so that the sum of all numbers in one group is equal to that in the other. Prove that one can remove two numbers from each group so that the sums of all numbers in each group are still the same.

(b) The numbers $1, 2, ..., n$ are divided into two groups so that the sum of all numbers in one group is equal to that in the other. Is it true that for every such $n > 4$ one can remove two numbers from each group so that the sums of all numbers in each group are still the same?

**2.** The sides $AB$ and $AC$ are tangent at points $P$ and $Q$ respectively to the incircle of triangle $ABC$. $R$ and $S$ are the midpoints of the sides $AC$ and $BC$ respectively and $T$ is the intersection point of the lines $PQ$ and $RS$. Prove that $T$ lies on the bisector of $\angle ABC$.

**3.** In a chess tournament every two participants play each other exactly once. A win is worth one point, a draw is worth half a point and a loss is worth zero points. Looking back at the end of the tournament a game is called an upset if the total number of points obtained by the winner of that game is less than the total number of points obtained by the loser of that game.

(a) Prove that the number of upsets is always strictly less than three-quarters of the total number of games in the tournament.

(b) Prove that three-quarters cannot be replaced by a smaller number.

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IPS 4. - 03.01.13.

**1.** Let $n$ be an odd integer greater than 1, and let $k_1, k_2, \ldots, k_n$ be given integers. For each of the $n!$ permutations $a = (a_1, a_2, \ldots, a_n)$ of $1, 2, ..., n$, let

$$S(a) = \sum_{i=1}^{n} k_i a_i$$

Prove that there are two permutations $b$ and $c$, $b \neq c$, such that $n!$ is a divisor of $S(b) - S(c)$.

**2.** The chords $AC$ and $BD$ of a circle with centre $O$ intersect at the point $K$. The circumcentres of triangles $AKB$ and $CKD$ are $M$ and $N$ respectively. Prove that $OM = KN$.

**3.** $n$ diameters divide a disk into $2n$ equal sectors, $n$ of the sectors are coloured blue, and the other $n$ are coloured red (in arbitrary order). Blue sectors are numbered from 1 to $n$ in the anticlockwise direction, starting from an arbitrary blue sector. Red sectors are numbered from 1 to $n$ in the clockwise direction, starting from an arbitrary red sector. Prove that there is a semi-disk containing sectors with all numbers from 1 to $n$.