Graphs, decompositions, geometric background

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Problem 1. Given a complete graph on $n = 101$ vertices, whose edges are coloured by t colours. For every triple, either the induced 3 edges are coloured differently (with 3 colours), or they have the same colour.
a) Prove that $t = 1$ or $t \geq 10$.
b) Give accurate bound for the number of colours (t) for every given n.
c) Find constructions (right colourings) where t is small but $t > 1$.

Problem 2.
a) In a table tennis competition of Hungland, two teams compete the following way: each of their $n - n$ player plays (once) against every player in the opponent team. How many rounds does it take to organize this event, if there are enough ($\geq n$) tables in the sport center?
b) How may ways can the organizer schedule the matches, if he wants to do it with minimum number of rounds? Give both upper and lower bounds, try to make the gap as small as possible. (** warning: no exact answer is known in general)

Problem 3. The rules of the table tennis competition are amended: each of their $n - n$ player plays (once) against exactly k players in the opponent team ($k < n$). How many rounds does it take to organize this event? Can the opponents be assigned in such a way that more than k rounds are necessary?

Problem 4. How many matches can be scheduled between the two team, if there do not exist two pairs from different teams, who played all their possible 4 matches?

Problem 5. A colouring of (the vertex set of) a graph G is proper, if every edge connects two differently coloured vertices. The graph is *definitely* colourable with k colours if any pair u and v have either the same colour for every proper k-colouring, or different colours for every proper k colouring. Prove that if G is *definitely* colourable with 3 colours and has $n > 3$ vertices, then G has at least $2n - 3$ edges.

Problem 6. We have a chessboard with 7 rows and 7 columns. A $k \times l$ sub-board of our chessboard is determined by the intersections of an arbitrary set of k rows and l coloums. How many chessmen can be placed at least on our chessboard, if none of the 2×3 and the 3×2 subboards of our chessboard is empty? Try to give lower and upper bounds!

Problem 7. Prove that if a set S intersects every line of a projective plane of order q (containing $q + 1$ lines through a point, $q + 1$ points on a line) then S contains at least $q + \sqrt{q} + 1$ points, otherwise it contains a line.

Problem 8. The fairy's favorite complete directed graph G on $|V(G)| = 2013$ vertices was stolen by an evil goblin. (That is, either $\vec{uv}$ or $\vec{vu}$ is in the edge set $E(G)$.) Instead of giving it back, the goblin presents 2013 other complete directed graph constructed the following way. He picks a vertex of G, and switches the orientations of all incident edges. (Of course he does not reveal which vertex he chose at a particular modified digraph.) Can the fairy reconstruct her original digraph?