The isogonal conjugate

Problems from András Hraskó, lecture by Zoltán Gyenes

Problem 1. Point P is given inside triangle ABC. Reflect line AP with respect to the inner (or outer) angle bisector at vertex A, line BP with respect to the inner angle bisector at vertex B, finally line CP with respect to the inner angle bisector at vertex C. Prove the three lines we got as a result also concurrent (goes through a point). (This new common point is called the isogonal conjugate of point P.)

What if P is not inside the triangle?

Problem 2.
What is the isogonal conjugate of the following points:
a) the incenter,
b) the centers of the circles that touches three sides of a triangle,
c) the center of the circumscribed circle,
d) the orthocenter?

Problem 3. Take a point P inside triangle ABC and project orthogonally P onto the sides of the triangle. Connect the feet of the perpendiculars on side AB and AC, and draw the perpendicular to this line segment from vertex A, the resulting line is a'. We get b' and c' similarly. Prove that these three lines are concurrent.

Problem 4. A circle intersects each of the three sides of a triangle in two points.

a) Prove that if we choose one of the points of intersection on each side (A_1, B_1, C_1), and the perpendiculars drawn from each are concurrent (at pont P_1), then taking the other three points of intersection (A_2, B_2, C_2), the perpendiculars drawn from these are also concurrent (at P_2).

Prove that in this situation

b) $ABP_1\sphericalangle \equiv P_2BC\sphericalangle$, $\quad BCP_1\sphericalangle \equiv P_2CA\sphericalangle$, $\quad CAP_1\sphericalangle \equiv P_2AB\sphericalangle \pmod{180^\circ}$!

c) $\frac{P_1A_1}{P_1C_1} = \frac{P_2C_2}{P_2A_2}$, $\quad \frac{P_1B_1}{P_1A_1} = \frac{P_2A_2}{P_2B_2}$, $\quad \frac{P_1C_1}{P_1B_1} = \frac{P_2B_2}{P_2C_2}$!

Problem 5. Triangle ABC is given in the plane. Take
– circle k_1, which passes through A and tangent to side BC at point B,
– circle k_2, which passes through B and tangent to side CA at point C,
– circle k_3, which passes through C and tangent to side AB at point A.
Prove that these three circles have a common point.

Problem 6. Prove that there is exactly one such point Q inside triangle ABC, for which $ACQ\angle = CBQ\angle = BAQ\angle$.

Prove that there is exactly one such point R inside triangle ABC, for which $CAR\angle = BCR\angle = ABR\angle$.

Definition. These two points are called the *Brocard-points* of the triangle.

Problem 7.
Take the two Brocard points of a triangle, i.e. points Q and R appearing on problems 5. and 6. Prove that the angles $ACQ\angle$ and $BCR\angle$ are equal. Shortly: the two Brocard-angles associated with the two Brocard-points are equal.

Problem 8.* *Malfatti's theorem*
Take three oriented circles: k_1, k_2, k_3. The oriented lines e_3, f_3 are tangent to k_1 and antitangent to k_2-t. Lines e_2, f_2 are tangent to k_3 and antitangent to k_1, and lines e_1 and f_1 are tangent to k_2-t and antitangent to k_3. Prove that e_1, e_2 and e_3 are concurrent iff f_1, f_2 and f_3 are concurrent.

Problem 9.* Quadrilateral $ABCD$ has an inscribed circle, and the line e through its vertex A intersects side BC in M, the extension of side CD in N. Denote the incenters of triangles ABM, MCN, NDA by I_1, I_2 and I_3 respectively. Show that the orthocenter of triangle $I_1I_2I_3$ is on line e!

Problem 10. A circle passing through vertices B and C of triangle ABC intersects the lines of side AB, AC in points C', B'. Prove that the midpoint of line segment $B'C'$ moves on a line by changing the circle.

Definition. This line is called the *symmedian* from A.

Problem 11. Prove that the three symmedians are concurrent.

Definition. The resulting point is called the *Lemoine-Grebe point* of the triangle.

Problem 12. We draw the circumscribed circle of triangle ABC and the tangents at points B and C, which intersect in point S. Prove that point S is on the symmedian from A.

Problem 13. Let L denote the Lemoine-Grebe point of triangle ABC. Draw the three antiparallels to the sides through point L. Prove that the six endpoints of these three line segments are on a circle. What is the center of this circle?

Problem 14.* Let us draw parallels through the Lemoine-Grebe point of triangle ABC. Prove that the six points of intersections of these three lines are on a circle.

Problem 15.* (IMO 1996) Point P inside triangle ABC satisfies

$$\sphericalangle APB - \sphericalangle ACB = \sphericalangle APC - \sphericalangle ABC.$$

Let D and E the incenters of triangle APB and APC respectively. Prove that lines AP, BD and CE are concurrent.