Tata, 28 December 2012

1. We are given four points on a line in the following order: A, B, C, D. Moreover, we know that $AB = CD$. Is it possible to construct the midpoint of segment BC, if we are only allowed to use a straightedge?
(KöMaL N. 166., February 1998)

2. In an acute triangle ABC, $\alpha < \beta$ (with conventional notations). Let R and P be the feet of the altitudes drawn from vertices A and C, respectively. Let Q denote a point of line AB, different from P, such that $AP \cdot BQ = AQ \cdot BP$. Prove that line RB bisects the angle PRQ. (KöMaL B. 4477., October 2012; proposed by József Mészáros)

3. Let M be the midpoint of a chord PQ of a circle, through which two other chords AB and CD are drawn; AD cuts PQ at X and BC cuts PQ at Y. Prove that M is also the midpoint of XY. (Butterfly problem; William George Horner, 1815)

4. The incircle of a triangle ABC is tangent to the sides BC, AC and AB at the points A_1, B_1 and C_1, respectively. Let F denote the midpoint of the line segment A_1B_1. Prove that $\angle B_1C_1C = \angle A_1C_1F$.

5. The escribed circle drawn to side AC of a triangle ABC is tangent to the lines of sides BC, AC and AB at the points A_1, B_1 and C_1, respectively. Let F denote the midpoint of the line segment A_1B_1. Prove that $\angle B_1C_1C = \angle A_1C_1F$. (KöMaL A. 564., April 2012)

6. In a quadrilateral $ABCD$, let $P = AC \cap BD$, $I = AD \cap BC$, and let Q be an arbitrary point which is not collinear with any two of points A, B, C, D. Then $\angle AQD = \angle CQB$ if and only if $\angle BQP = \angle IQA$. (IMO Shortlist 2007; comment on problem G3)

7. The diagonals of a trapezoid $ABCD$ intersect at point P. Point Q lies between the parallel lines BC and AD such that $\angle AQD = \angle CQB$, and line CD separates points P and Q. Prove that $\angle BQP = \angle DAQ$. (IMO Shortlist 2007/G3; proposed by Ukraine)

8. We are given six points on a line ℓ in the following order: A, B, C, D, E, F. Moreover, we know that $AB = CD = EF$. Is it possible to construct a line parallel with ℓ, if we are only allowed to use a straightedge?

9. A circle k with center O and four distinct fixed points A, B, C, D lying on it are given. The circle k' intersects k perpendicularly at A and B. Let X be a variable point on the line OA. Let U, other than A, be the second intersection of the circles ACX and k'. Let V, other than A, be the second intersection of the circles ADX and k'. Let W, other than B, be the second intersection of the circle BDU and the line OB. Finally, let E, other than B, be the second intersection of the circles BVW and k. Prove that the location of the point E is independent from the choice of the point X. (KöMaL A. 544., October 2011)

10. A tetrahedron $OA_1A_2A_3$ is given. For $i = 1, 2, 3$ let B_i be a point in the interior of the edge OA_i and let C_i be a point on the ray OA_i, beyond A_i. Suppose that the polyhedron bounded by the six planes $OA_{i+1}A_{i+2}$ and $B_iA_{i+1}A_{i+2}$ $(i = 1, 2, 3)$ circumscribes a sphere, and the polyhedron bounded by the planes $B_iA_{i+1}A_{i+2}$ and $C_iA_{i+1}A_{i+2}$ also circumscribes a sphere. Prove that the polyhedron bounded by the planes $OA_{i+1}A_{i+2}$ and $C_iA_{i+1}A_{i+2}$ also circumscribes a sphere. (KöMaL A. 547., November 2011)

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11. Let a and b be two projective lines which are tangents to a circle ω at A and B, respectively and let $C = a \cap b$. Define a map $f : a \to b$ as follows. Let $f(A) = C$, $f(C) = B$. For any other point $P \in a$, let $f(P) \in b$ be the point for which the line $(P, f(P))$ is tangent to ω. Prove that f preserves cross-ratio.

12. In the triangle ABC, denote by A_1, B_1, and C_1 the feet of altitudes. Let P be the perpendicular foot of C_1 on line A_1B_1, and let Q be that point of line A_1B_1 for which $AQ = BQ$. Show that $\angle PAQ = \angle PBQ = \angle PC_1C$. (KöMaL A. 470. January 2009)

13. Given an acute triangle ABC with angles α, β and γ at vertices A, B and C, respectively, such that $\beta > \gamma$. Point I is the incenter, and R is the circumradius. Point D is the foot of the altitude from vertex A. Point K lies on line AD such that $AK = 2R$, and D separates A and K. Finally, lines DI and KI meet sides AC and BC at E and F, respectively.

Prove that if $IE = IF$ then $\beta \leq 3\gamma$. (IMO Shortlist 2007/G7; proposed by Iran)

Hint: Prove that $\angle KID = \frac{\beta-\gamma}{2}$.

14. Two circles k_1 and k_2 with centres O_1 and O_2, respectively, intersect perpendicularly at P and Q. Their external homothety center is H. The line t is tangent to k_1 at T_1 and tangent to k_2 at T_2. Let X be a point in the interior of the two circles such that $HX = HP = HQ$, and let X' be the reflection of X about t. Let the circle $XX'T_2$ and the shorter arc PQ of k_1 meet at U_1, and let the circle $XX'T_1$ and the shorter arc PQ of k_2 meet at U_2. Finally, let the lines O_1U_1 and O_2U_2 meet at V. Show that $VU_1 = VU_2$. (KöMaL A. 572., November 2012)

15. Let k be the incircle in the triangle ABC, which is tangent to the sides AB, BC, CA at the points C_0, A_0 and B_0, respectively. The angle bisector starting at A meets k at A_1 and A_2, the angle bisector starting at B meets k at B_1 and B_2; $AA_1 < AA_2$ and $BB_1 < BB_2$. The circle $k_1 \neq k$ is tangent externally to the side CA at B_0 and it is tangent to the line AB. The circle $k_2 \neq k$ is tangent externally to the side BC at A_0, and it is tangent to the line AB. The circle k_3 is tangent to k at A_1, and it is tangent to k_1 at point P. The circle k_4 is tangent to k at B_1, and it is tangent to k_2 at point Q. Prove that the radical axis between the circles A_1A_2P and B_1B_2Q is the angle bisector starting at C. (KöMaL A. 564., May 2012)

16. There is given a circle k in the plane, a chord AB of k, furthermore four interior points, C, D, E and F, on the line segment AB. Draw an arbitrary chord X_1X_2 of k through point C, a chord Y_1Y_2 through D, a chord U_1U_2 through E, finally a chord V_1V_2 through F in such a way that X_1, Y_1, U_1 and V_1 lie on the same side of the line AB, and

$$\frac{AX_1 \cdot BX_2}{X_1X_2} = \frac{AY_2 \cdot BY_1}{Y_1Y_2} = \frac{AU_1 \cdot BU_2}{U_1U_2} = \frac{AV_2 \cdot BV_1}{V_1V_2}$$

holds. Let Z be the intersection of the lines X_1X_2 and Y_1Y_2, and let W be the intersection of U_1U_2 and V_1V_2. Show that the lines ZW obtained in this way are concurrent or they are parallel to each other. (KöMaL A. 529., February 2011)

17. Given a triangle ABC. For an arbitrary interior point X of the triangle denote by $A_1(X)$ the point intersection of the lines AX and BC, denote by $B_1(X)$ the point intersection of the lines BX and CA, and denote by $C_1(X)$ the point intersection of the lines CX and AB. Construct such a point P in the interior of the triangle for which each of the quadrilaterals $AC_1(P)PB_1(P)$, $BA_1(P)PC_1(P)$ and $CB_1(P)PA_1(P)$ has an inscribed circle. (KöMaL A. 570., November 2012; proposed by Gábor Holló)