Positive polynomials and sums of squares

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Let us begin with a simple observation. For all real numbers x and y, we have

$$\frac{x^2+y^2}{2} - \left(\frac{x+y}{2}\right)^2 = \left(\frac{x-y}{2}\right)^2 \geq 0.$$

We shall now explore how this phenomenon generalizes. We need a definition: a *polynomial* in the variables $x_1, \ldots, x_n$ is an expression that can be built from these variables and real numbers using addition, subtraction and multiplication. Note that subtraction is not really needed since $f - g = f + (-1)g$.

For example, the expression on the left hand side above is a polynomial in x and y, and we see that it can be rewritten as the square of another polynomial.

Here are some questions.

1. (a) Can

   $$\frac{x^2+y^2+z^2}{3} - \left(\frac{x+y+z}{3}\right)^2$$

   be written as the square of a polynomial?

   (b) Can it be written as a sum of squares of polynomials?

2. Can

   $$\frac{x_1^2+\cdots+x_n^2}{n} - \left(\frac{x_1+\cdots+x_n}{n}\right)^2$$

   be written as a sum of squares of polynomials?

3. Same question for

   $$\frac{x^3+y^3}{2} - \left(\frac{x+y}{2}\right)^3.$$

4. Can the last polynomial be obtained by addition and multiplication (but no subtraction) from x, y and squares of suitable polynomials?

5. Can the polynomial

   $$\frac{x_1^p+\cdots+x_n^p}{n} - \left(\frac{x_1+\cdots+x_n}{n}\right)^p$$

   be obtained by addition and multiplication from $x_1, \ldots, x_n$ and squares of polynomials? (Note that p is a natural number.)

6. Can the polynomial

   $$\left(\frac{x+y+z}{3}\right)^3 - xyz$$

   be obtained by addition and multiplication from x, y, z and squares of polynomials?

7. Suppose that the polynomial $f(x_1, \ldots, x_n)$ is non-negative for all non-negative x_1, x_2, $\ldots$, x_n (as in all examples above). Does it follow that f can be obtained by addition and multiplication from x_1, x_2, $\ldots$, x_n and squares of suitable polynomials?